

DUAL π -RICKART MODULES

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ABSTRACT. Let R be an arbitrary ring with identity and M a right R -module with $S = \text{End}_R(M)$. In this paper we introduce dual π -Rickart modules as a generalization of π -regular rings as well as that of dual Rickart modules. The module M is said to be *dual π -Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $\text{Im} f^n = eM$. We prove that some results of dual Rickart modules can be extended to dual π -Rickart modules for this general settings. We investigate relations between a dual π -Rickart module and its endomorphism ring.

2010 MSC: 13C99, 16D80, 16U80

Key words: π -Rickart modules, dual π -Rickart modules, Fitting modules, generalized left principally projective rings, π -regular rings.

1. INTRODUCTION

Throughout this paper R denotes an associative ring with identity, and modules are unitary right R -modules. For a module M , $S = \text{End}_R(M)$ is the ring of all right R -module endomorphisms of M . In this work, for the (S, R) -bimodule M , $l_S(\cdot)$ and $r_M(\cdot)$ are the left annihilator of a subset of M in S and the right annihilator of a subset of S in M , respectively. A ring is *reduced* if it has no nonzero nilpotent elements. *Baer rings* [8] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. Principally projective rings were introduced by Hattori [3] to study the torsion theory, that is, a ring R is called *left (right) principally projective* if every principal left (right) ideal is projective. The concept of left (right) principally projective rings (or left (right) Rickart rings) has been comprehensively studied in the literature. Regarding a generalization of Baer rings as well as principally projective rings, recall that a ring R is called *generalized left (right) principally projective* if for any $x \in R$, the left (right) annihilator of x^n is generated by an idempotent for some positive integer n . A number of papers have been written on generalized principally

projective rings (see [4] and [7]). A ring R is (*von Neumann*) *regular* if for any $a \in R$ there exists $b \in R$ with $a = aba$. The ring R is called *π -regular* if for each $a \in R$ there exist a positive integer n and an element x in R such that $a^n = a^n x a^n$. Similarly, call a ring R *strongly π -regular* if for every element $a \in R$ there exist a positive integer n (depending on a) and an element $x \in R$ such that $a^n = a^{n+1}x$, equivalently, there exists $y \in R$ such that $a^n = ya^{n+1}$. Every regular ring is π -regular and every strongly π -regular ring is π -regular. There are regular or π -regular rings which are not strongly π -regular.

According to Rizvi and Roman, a module M is said to be *Rickart* [10] if for any $f \in S$, $r_M(f) = eM$ for some $e^2 = e \in S$. The class of Rickart modules is studied extensively by different authors (see [1] and [11]). Recently the concept of a Rickart module is generalized in [16] by the present authors. The module M is called *π -Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $r_M(f^n) = eM$. Dual Rickart modules are defined by Lee, Rizvi and Roman in [12]. The module M is called *dual Rickart* if for any $f \in S$, $\text{Im} f = eM$ for some $e^2 = e \in S$.

In the second section, we investigate general properties of dual π -Rickart modules and Section 3 contains the results on the structure of endomorphism ring of a dual π -Rickart module. In what follows, we denote by \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{Z}_n integers, rational numbers, real numbers and the ring of integers modulo n , respectively, and $J(R)$ denotes the Jacobson radical of a ring R .

2. DUAL π -RICKART MODULES

In this section, we introduce the concept of a dual π -Rickart module that generalizes the notion of a dual Rickart module as well as that of a π -regular ring. We prove that some properties of dual Rickart modules hold for this general setting. Although every direct summand of a dual π -Rickart module is dual π -Rickart, a direct sum of dual π -Rickart modules is not dual π -Rickart. We give an example to show that a direct sum of dual π -Rickart modules may not be dual π -Rickart. It is shown that the class of some abelian dual π -Rickart modules is closed under direct sums.

We start with our main definition.

Definition 2.1. Let M be an R -module with $S = \text{End}_R(M)$. The module M is called *dual π -Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $\text{Im} f^n = eM$.

For the sake of brevity, in the sequel, S will stand for the endomorphism ring of the module M considered. Dual π -Rickart modules are abundant around. Every semisimple module, every injective module over a right hereditary ring and every module of finite length are dual π -Rickart. Also every quasi-projective strongly co-Hopfian module, every quasi-injective strongly Hopfian module, every Artinian and Noetherian module is dual π -Rickart (see Corollary 2.19). Every finitely generated module over a right Artinian ring is a dual π -Rickart module (see Proposition 2.20).

Proposition 2.2. *Let R be a ring. Then the right R -module R is a dual π -Rickart module if and only if R is a π -regular ring.*

Proof. If the right R -module R is a dual π -Rickart module and $f \in R$, then there exist $e^2 = e \in R$ and a positive integer n such that $\text{Im} f^n = eR$. There exist $x, y \in R$ such that $e = f^n x$ and $f^n = ey$. Multiplying the first equation from the right by f^n , we have $f^n x f^n = ey = f^n$. Conversely, assume that R is a π -regular ring. Let $g \in R$. Then there exist a positive integer n and $x \in R$ such that $g^n = g^n x g^n$. Hence $e = g^n x$ is an idempotent of R . Since $e \in g^n R$ and $g^n = g^n x g^n = e g^n \in eR$, we have $\text{Im} g^n = eR$. Therefore the right R -module R is dual π -Rickart. \square

It is clear that every dual Rickart module is dual π -Rickart. The following example shows that every dual π -Rickart module need not be dual Rickart.

Example 2.3. Let R denote the ring $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ and M the right R -module $\begin{pmatrix} 0 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ with usual matrix operations. If $f \in S = \text{End}_R(M)$, then there exist $a, b, c \in \mathbb{Z}_2$ such that

$$f \begin{pmatrix} 0 & x \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & ax \\ by & cx + bz \end{pmatrix}$$

By using this image of f , we prove that there exists a positive integer n such that $\text{Im} f^n$ is a direct summand of M . Consider the following cases for $a, b, c \in \mathbb{Z}_2$.

Case 1. If $a = b = c = 1$, then f is an epimorphism.

Case 2. If $a = 0, b = 0, c = 1$, then $f^2 = 0$.

Case 3. If $a = 0, b = 1, c = 1$ or $a = 0, b = 1, c = 0$, then in either case

$\text{Im} f = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \mid x, y \in \mathbb{Z}_2 \right\}$ is a direct summand of M .

Case 4. If $a = 1, b = 0, c = 1$, then $\text{Im}f = \left\{ \begin{pmatrix} 0 & x \\ 0 & x \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\}$ is a direct summand of M .

Case 5. If $a = 1, b = 0, c = 0$, then $\text{Im}f = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\}$ is a direct summand of M .

Case 6. If $a = 1, b = 1, c = 0$, then f is an identity map.

Case 7. If $a = 0, b = 0, c = 0$, then f is a zero map.

In all cases there exists a positive integer n such that $\text{Im}f^n$ is a direct summand of M and so M is a dual π -Rickart module. The module M is not dual Rickart by the second case, since $\text{Im}f = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\}$.

Our next aim is to find conditions under which a dual π -Rickart module is dual Rickart.

Proposition 2.4. *Let M be a dual Rickart module. Then M is dual π -Rickart. The converse holds if S is a reduced ring.*

Proof. The first statement is clear. Suppose that S is a reduced ring and M is a dual π -Rickart module. Let $f \in S$. There exist a positive integer n and an idempotent $e \in S$ such that $\text{Im}f^n = eM$. If $n = 1$, there is nothing to do. Assume that $n > 1$. Then $(1 - e)f^n M = 0$ and so $(1 - e)f^n = 0$. Since S is a reduced ring, e is central and $((1 - e)f)^n = 0$. Also it implies $(1 - e)f = 0$ or $f = ef$. Thus $\text{Im}f \leq eM$. The reverse inclusion $eM \leq \text{Im}f$ follows from $eM = f^n M \leq f(f^{n-1})M \leq fM$. Therefore $eM = \text{Im}f$ and M is a dual Rickart module. \square

By using a different condition on an endomorphism ring of a module we show that a dual π -Rickart module is dual Rickart. To do this we need the following lemma.

Lemma 2.5. *Let M be a module. Then M is dual π -Rickart and S is a domain if and only if every nonzero element of S is an epimorphism.*

Proof. The sufficiency is clear. For the necessity, let M be a dual π -Rickart module and $0 \neq f \in S$. Then there exist a positive integer n and an idempotent $e \in S$ such that $\text{Im}f^n = eM$. Hence $f^n = ef^n$. Since S is a domain and f^n is nonzero, we have $e = 1$ and so $\text{Im}f^n = M$. This implies that $\text{Im}f = M$. Thus f is an epimorphism. \square

Recall that a module M has C_2 condition if any submodule N of M which is isomorphic to a direct summand of M is a direct summand, while a module M is said to have D_2 condition if any submodule N of M with M/N isomorphic to a direct summand of M , then N is a direct summand of M . In the next result we obtain relations between π -Rickart and dual π -Rickart modules by using C_2 and D_2 conditions. An endomorphism f of a module M is called *morphic* [9] if $M/fM \cong \text{Ker} f$. The module M is called *morphic* if every endomorphism of M is morphic.

Theorem 2.6. *Let M be a module. Then we have the following.*

- (1) *If M is a dual π -Rickart module with D_2 condition, then it is π -Rickart.*
- (2) *If M is a π -Rickart module with C_2 condition, then it is dual π -Rickart.*
- (3) *If M is projective morphic, then it is π -Rickart if and only if it is dual π -Rickart.*

Proof. Since $M/\text{Ker} f^n \cong \text{Im} f^n$, D_2 and C_2 conditions complete the proof of (1) and (2). The proof of (3) is clear. \square

The next result is an immediate consequence of Theorem 2.6.

Corollary 2.7. *Let M be a module with C_2 and D_2 conditions. Then M is a dual π -Rickart module if and only if it is π -Rickart.*

In [12, Proposition 2.6], it is shown that M is a dual Rickart module if and only if the short exact sequence $0 \rightarrow \text{Im} f \rightarrow M \rightarrow M/\text{Im} f \rightarrow 0$ splits for any $f \in S$. In this direction we can give a similar characterization for dual π -Rickart modules.

Lemma 2.8. *The following are equivalent for a module M .*

- (1) *M is a dual π -Rickart module.*
- (2) *For every $f \in S$ there exists a positive integer n such that the short exact sequence $0 \rightarrow \text{Im} f^n \rightarrow M \rightarrow M/\text{Im} f^n \rightarrow 0$ splits.*

Proof. For any $f \in S$ and any positive integer n consider the short exact sequence $0 \rightarrow \text{Im} f^n \rightarrow M \rightarrow M/\text{Im} f^n \rightarrow 0$. The short exact sequence splits in M if and only if $\text{Im} f^n$ is a direct summand of M if and only if M is a dual π -Rickart module. \square

One may suspect that every submodule of a dual π -Rickart module is dual π -Rickart. The following example shows that this is not the case.

Example 2.9. Consider \mathbb{Q} as a \mathbb{Z} -module. Then $S = \text{End}_{\mathbb{Z}}(\mathbb{Q})$ is isomorphic to \mathbb{Q} . Since every element of S is an isomorphism or zero, \mathbb{Q} is dual π -Rickart. Now consider the submodule \mathbb{Z} and $f \in \text{End}_{\mathbb{Z}}(\mathbb{Z})$ defined by $f(x) = 2x$, where $x \in \mathbb{Z}$. Since the image of any power of f can not be a direct summand of \mathbb{Z} , the submodule \mathbb{Z} is not dual π -Rickart.

Although every submodule of a dual π -Rickart module need not be dual π -Rickart by Example 2.9, we now prove that every direct summand of dual π -Rickart modules is also dual π -Rickart.

Proposition 2.10. *Let M be a dual π -Rickart module. Then every direct summand of M is also dual π -Rickart.*

Proof. Let $M = N \oplus P$ with $S_N = \text{End}_R(N)$. Define $g = f \oplus 0|_P$, for any $f \in S_N$ and so $g \in S$. By hypothesis, there exist a positive integer n and $e^2 = e \in S$ such that $\text{Im} g^n = eM$ and $g^n = f^n \oplus 0|_P$. Hence $eM = \text{Im} g^n = f^n N \leq N$. Let $M = eM \oplus Q$ for some submodule Q . Thus $N = eM \oplus (N \cap Q) = f^n N \oplus (N \cap Q)$. Therefore N is dual π -Rickart. \square

Corollary 2.11. *Let R be a π -regular ring with $e = e^2 \in R$. Then $M = eR$ is a dual π -Rickart R -module.*

Here we give the following result for π -regular rings.

Corollary 2.12. *Let $R = R_1 \oplus R_2$ be a π -regular ring with direct sum of the rings R_1 and R_2 . Then the rings R_1 and R_2 are also π -regular.*

We now characterize π -regular rings in terms of dual π -Rickart modules.

Theorem 2.13. *Let R be a ring. Then R is π -regular if and only if every cyclic projective R -module is dual π -Rickart.*

Proof. The sufficiency is clear. For the necessity, let $M = mR$ be a projective module. Then $R = r_R(m) \oplus I$ for some right ideal I of R . Let $I \xrightarrow{\varphi} M$ denote the isomorphism and $f \in S$. By Proposition 2.2 and Proposition 2.10, $(\varphi^{-1}f\varphi)^n I = (\varphi^{-1}f^n\varphi)I$ is a direct summand of I for some positive integer n . Hence $I = (\varphi^{-1}f^n\varphi)I \oplus K$ for some right ideal K of I . Thus $\varphi I = (f^n\varphi)I \oplus \varphi K$, and so $M = f^n M \oplus \varphi K$. Therefore M is dual π -Rickart. \square

Theorem 2.14. *Let R be a ring and consider the following conditions.*

- (1) *Every free R -module is dual π -Rickart.*
- (2) *Every projective R -module is dual π -Rickart.*
- (3) *Every flat R -module is dual π -Rickart.*

Then (3) \Rightarrow (2) \Leftrightarrow (1). Moreover (2) \Rightarrow (3) holds for finitely presented modules.

Proof. (3) \Rightarrow (2) \Rightarrow (1) Clear. (1) \Rightarrow (2) Let M be a projective R -module. Then M is a direct summand of a free R -module F . By (1), F is dual π -Rickart, and so is M due to Proposition 2.10.

(2) \Rightarrow (3) is clear from the fact that finitely presented flat modules are projective. \square

The next example reveals that a direct sum of dual π -Rickart modules need not be dual π -Rickart.

Example 2.15. Let R denote the ring $\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$ and M the R -module $\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$. Let $f \in S$. Then there exist $a, c, u, t \in R$ such that $f \begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} ax + ur & ay + us \\ cx + tr & cy + ts \end{pmatrix}$ where $\begin{pmatrix} x & y \\ r & s \end{pmatrix} \in M$. Consider $f \in S$ defined by $a = c = 0$, $u = 3$ and $t = 2$. This implies that $f \begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} 3r & 3s \\ 2r & 2s \end{pmatrix}$ and for any positive integer n we obtain

$$f^n \begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} 3(2^{n-1})r & 3(2^{n-1})s \\ 2^n r & 2^n s \end{pmatrix}$$

It follows that $f^n M$ can not be a direct summand. On the other hand, consider the submodules $N = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ of M . Then $\text{End}_R(N)$ and $\text{End}_R(K)$ are isomorphic to \mathbb{R} . Hence N and K are dual π -Rickart modules but M is not dual π -Rickart.

The following lemma is useful to show that a direct sum of some dual π -Rickart modules is dual π -Rickart.

Lemma 2.16. *Let M be a module and $f \in S$. If $\text{Im} f^n = eM$ for some central idempotent $e \in S$ and a positive integer n , then $\text{Im} f^{n+1} = eM$.*

Proof. Let $f \in S$ and $\text{Im} f^n = eM$ for some central idempotent $e \in S$ and a positive integer n . It is clear that $\text{Im} f^{n+1} \subseteq \text{Im} f^n$. Let $f^n(x) \in \text{Im} f^n$, then $f^n(x) = ef^n(x) = f^n e(x)$. Since $e(x) \in \text{Im} f^n$, $e(x) = f^n(y)$ for some $y \in M$. So $f^n(x) = f^n(f^n(y)) = f^{n+1}(f^{n-1}(y)) \in \text{Im} f^{n+1}$. This completes the proof. \square

A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any $a, e^2 = e \in R$. A module M is called *abelian* [14] if $fem = efm$ for any $f \in S$, $e^2 = e \in S$, $m \in M$. Note that M is an abelian module if and only if S is an abelian ring. We now prove that a direct sum of dual π -Rickart modules is dual π -Rickart for some abelian modules.

Proposition 2.17. *Let M_1 and M_2 be abelian R -modules. If M_1 and M_2 are dual π -Rickart with $\text{Hom}_R(M_i, M_j) = 0$ for $i \neq j$, then $M_1 \oplus M_2$ is a dual π -Rickart module.*

Proof. Let $S_1 = \text{End}_R(M_1)$, $S_2 = \text{End}_R(M_2)$ and $M = M_1 \oplus M_2$. We may describe S as $\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$. Let $\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \in S$, where $f_1 \in S_1$ and $f_2 \in S_2$. Then there exist positive integers n, m and $e_1^2 = e_1 \in S_1$ and $e_2^2 = e_2 \in S_2$ such that $\text{Im} f_1^n = e_1 M_1$ and $\text{Im} f_2^m = e_2 M_2$. Consider the following cases:

- (i) Let $n = m$. Obviously, $\text{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^n = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M$.
- (ii) Let $n < m$. By Lemma 2.16, we have $\text{Im} f_1^n = \text{Im} f_1^m = e_1 M_1$. Clearly, $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M \leq \text{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m$. Now let $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \text{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m$. Then $m_1 \in \text{Im} f_1^m = e_1 M_1$ and $m_2 \in \text{Im} f_2^m = e_2 M_2$. Hence $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$. Thus $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M$. Therefore $\text{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m \leq \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M$.
- (iii) Let $m < n$. Since M_2 is abelian, the proof is similar to case (ii). \square

We close this section with the relations among strongly co-Hopfian modules, Fitting modules and dual π -Rickart modules.

Recall that a module M is called *co-Hopfian* if every injective endomorphism of M is an automorphism, while M is called *strongly co-Hopfian* [5], if for any endomorphism f of M the descending chain

$$\text{Im} f \supseteq \text{Im} f^2 \supseteq \cdots \supseteq \text{Im} f^n \supseteq \cdots$$

stabilizes.

We now give a relation between abelian and strongly co-Hopfian modules by using dual π -Rickart modules.

Corollary 2.18. *Let M be a dual π -Rickart module and S an abelian ring. Then M is strongly co-Hopfian.*

Proof. It follows from Lemma 2.16 and [5, Proposition 2.6]. \square

A module M is said to be a *Fitting module* [5] if for any $f \in S$, there exists an integer $n \geq 1$ such that $M = \text{Ker} f^n \oplus \text{Im} f^n$. Due to Armendariz, Fisher and Snider [2] or [15, Proposition 5.7], the module M is Fitting if and only if S is strongly π -regular.

We now give the following relation between Fitting modules and dual π -Rickart modules.

Corollary 2.19. *Every Fitting module is a dual π -Rickart module.*

Then we have the following result.

Proposition 2.20. *Let R be an Artinian ring. Then every finitely generated R -module is dual π -Rickart.*

Proof. Let M be a finitely generated R -module. Then M is an Artinian and Noetherian module. Hence M is a Fitting module and so it is dual π -Rickart. \square

Proposition 2.21. *Let R be a ring and n a positive integer. If the matrix ring $M_n(R)$ is strongly π -regular, then R^n is a dual π -Rickart R -module.*

Proof. Let $M_n(R)$ be a strongly π -regular ring. Then by [5, Corollary 3.6], R^n is a Fitting R -module and so it is dual π -Rickart. \square

3. THE ENDOMORPHISM RING OF A DUAL π -RICKART MODULE

In this section we study relations between a dual π -Rickart module and its endomorphism ring. We prove that the endomorphism ring of a dual π -Rickart module is always a generalized left principally projective ring, the converse holds if the module is self-cogenerator. The modules whose endomorphism rings are π -regular are characterized. It is shown that if the module satisfies D_2 condition, then it is dual π -Rickart if and only if the endomorphism ring of the module is a π -regular ring.

Lemma 3.1. *If M is a dual π -Rickart module, then S is a generalized left principally projective ring.*

Proof. Let $f \in S$. By assumption, there exist $e^2 = e \in S$ and a positive integer n such that $\text{Im} f^n = eM$. Hence $l_S(f^n M) = S(1 - e) = l_S(f^n)$. Thus S is a generalized left principally projective ring. \square

The next result is a consequence of Theorem 2.10 and Lemma 3.1.

Corollary 3.2. *If R is a π -regular ring, then eRe is a generalized left principally projective ring for any $e^2 = e \in R$.*

Corollary 3.3. *Let M be a dual π -Rickart module and $f \in S$. Then Sf^n is a projective left S -module for some positive integer n .*

Proof. Clear from Lemma 3.1, since $Sf^n \cong S/l_S(f^n)$. \square

Recall that a module is called *self-cogenerator* if it cogenerates all its factor modules. The following result shows that the converse of Lemma 3.1 is true for self-cogenerator modules. On the other hand, Theorem 3.4 generalizes the result [17, 39.11].

Theorem 3.4. *Let M be a module and $f \in S$.*

- (1) *If Sf^n is a projective left S -module for some positive integer n , then the submodule $N = \bigcap \{ \text{Kerg} \mid g \in S, \text{Im} f^n \leq \text{Kerg} \}$ is a direct summand of M .*
- (2) *If M is self-cogenerator and S is a generalized left principally projective ring, then M is a dual π -Rickart module.*

Proof. (1) Let Sf^n be a projective left S -module for some positive integer n . Since $Sf^n \cong S/l_S(f^n)$, $l_S(f^n) = Se$ for some $e^2 = e \in S$. We prove $(1 - e)M = N$. Due to $ef^n M = 0$, we have $f^n M \leq (1 - e)M$. By definition of N we have $N \leq (1 - e)M$. Let $g \in S$ with $\text{Im} f^n \leq \text{Kerg}$. Then $gf^n M = 0$ or $gf^n = 0$. Hence $g \in l_S(f^n) = Se$ and $ge = g$. So $g(1 - e)M = 0$ from which we have $(1 - e)M \leq \text{Kerg}$ for all g with $\text{Im} f^n \leq \text{Kerg}$. Thus $(1 - e)M \leq N$. Therefore $(1 - e)M = N$.

(2) Assume that M is self-cogenerator and S is generalized left principally projective. There exist $e^2 = e \in S$, a positive integer n such that $l_S(f^n) = Se$ and $M/\text{Im} f^n$ is cogenerated by M . By [17, 14.5],

$$\bigcap \{ \text{Kerg} \mid g \in \text{Hom}(M/\text{Im} f^n, M) \} = 0.$$

Hence

$$\text{Im}f^n = \bigcap \{ \text{Ker}g \mid g \in S, \text{Im}f^n \leq \text{Ker}g \}.$$

Thus conditions of (1) are satisfied and so $\text{Im}f^n$ is a direct summand. \square

For an R -module M , it is shown that, if S is a von Neumann regular ring, then M is a dual Rickart module (see [12, Proposition 3.8]). We obtain a similar result for dual π -Rickart modules.

Lemma 3.5. *Let M be a module. If S is a π -regular ring, then M is dual π -Rickart.*

Proof. Let $f \in S$. Since S is π -regular, there exist a positive integer n and $g \in S$ such that $f^n = f^n g f^n$. Then $e = f^n g$ is an idempotent of S . Now we show that $\text{Im}f^n = f^n g M$. It is clear that $f^n M = e f^n M \leq e M$. For the other inclusion, let $m \in M$. Hence $em = f^n g m \in f^n M$. Thus $\text{Im}f^n = e M$. \square

Since every strongly π -regular ring is π -regular, we have the next result.

Corollary 3.6. *Let M be a module. If S is a strongly π -regular ring, then M is dual π -Rickart.*

The converse statement of Corollary 3.6 does not hold in general, that is, there exists a dual π -Rickart module having not a strongly π -regular endomorphism ring.

Example 3.7. Let D be a division ring, M a vector space over D with an infinite basis $\{e_i \in M \mid i = 1, 2, \dots\}$ and $S = \text{End}_D(M)$. As a semisimple right D -module, M is dual π -Rickart, and by [17, 3.9] S is a regular and so π -regular ring. Assume that S is a strongly π -regular ring and we reach a contradiction. Let $f \in S$ defined by $f(e_i) = e_{i+1}$ for all $i = 1, 2, 3, \dots$. By assumption, there is a positive integer n such that $f^n = f^{n+1}g$ for some $g \in S$. Then $f^n = f^{n+1}g$ implies $f^n S = f^{n+1}S$ and so $f^n M = f^{n+1}M$. Since $f^n(e_i) = e_{i+n}$ for all i , we have $f^n M = \sum_{i>n} e_i D \neq f^{n+1}M$. This is a contradiction. Hence S is not a strongly π -regular ring (see also [15, 5.5]).

The proof of Lemma 3.8 may be in the context.

Lemma 3.8. *Let M be a module. Then S is a π -regular ring if and only if there exists a positive integer n such that $\text{Ker}f^n$ and $\text{Im}f^n$ are direct summands of M for any $f \in S$.*

Now we recall some known facts that will be needed about π -regular rings.

Lemma 3.9. *Let R be a ring. Then*

- (1) *If R is π -regular, then eRe is also π -regular for any $e^2 = e \in R$.*
- (2) *If $M_n(R)$ is π -regular for any positive integer n , then so is R .*
- (3) *If R is a commutative ring, then R is π -regular if and only if $M_n(R)$ is π -regular for any positive integer n .*

Proposition 3.10. *Let R be a commutative π -regular ring. Then every finitely generated projective R -module is dual π -Rickart.*

Proof. Let M be a finitely generated projective R -module. So the endomorphism ring of M is $eM_n(R)e$ with some positive integer n and an idempotent e in $M_n(R)$. Since R is commutative π -regular, $M_n(R)$ is also π -regular, and so is $eM_n(R)e$ by Lemma 3.9. Hence M is dual π -Rickart by Lemma 3.5. \square

Theorem 3.11. *Let M be a module with D_2 condition. Then M is dual π -Rickart if and only if S is π -regular.*

Proof. The necessity holds by Lemma 3.5. For the sufficiency, let $0 \neq f \in S$. Since M is dual π -Rickart, $\text{Im} f^n$ is a direct summand of M for some positive integer n . Because of $M/\text{Ker} f^n \cong \text{Im} f^n$, D_2 condition implies that $\text{Ker} f^n$ is a direct summand of M . The rest is obvious from Lemma 3.8. \square

The following is a consequence of Proposition 3.10 and Theorem 3.11.

Corollary 3.12. *Let R be a commutative ring and satisfy D_2 condition. Then the following are equivalent.*

- (1) *R is a π -regular ring.*
- (2) *Every finitely generated projective R -module is dual π -Rickart.*

Recall that a module M is called *quasi-projective* if it is M -projective. Since every quasi-projective module has D_2 condition, we have the following.

Corollary 3.13. *If M is a quasi-projective dual π -Rickart module, then the endomorphism ring of M is a π -regular ring.*

Theorem 3.14. *The following are equivalent for a ring R .*

- (1) *$M_n(R)$ is π -regular for every positive integer n .*
- (2) *Every finitely generated projective R -module is dual π -Rickart.*

Proof. (1) \Rightarrow (2) Let M be a finitely generated projective R -module. Then $M \cong eR^n$ for some positive integer n and $e^2 = e \in M_n(R)$. Hence S is isomorphic to $eM_n(R)e$. By (1), S is π -regular. Thus M is π -Rickart due to Lemma 3.5.

(2) \Rightarrow (1) $M_n(R)$ can be viewed as the endomorphism ring of a projective R -module R^n for any positive integer n . By (2), R^n is dual π -Rickart. Then $M_n(R)$ is π -regular by Corollary 3.13. \square

Recall that an R -module M is called *duo* if every submodule of M is fully invariant, i.e., for any submodule N of M , $fN \leq N$ for each $f \in S$, equivalently, every right R -submodule of M is also left S -submodule. Our next aim is to determine to find conditions under which any factor module of a dual π -Rickart module is also dual π -Rickart.

Corollary 3.15. *Let M be a quasi-projective module and N a fully invariant submodule of M . If M is dual π -Rickart, then so is M/N .*

Proof. Let $f \in S$ and π denote the natural epimorphism from M to M/N . Consider the following diagram.

$$\begin{array}{ccc} M & \xrightarrow{\pi} & M/N \\ f \downarrow & & \downarrow f^* \\ M & \xrightarrow{\pi} & M/N \end{array}$$

Since N is fully invariant, we have $\text{Ker } \pi \subseteq \text{Ker } \pi f$. By the Factor Theorem, there exists a unique homomorphism f^* such that $f^* \pi = \pi f$. Hence we define a homomorphism $\varphi : S \rightarrow \text{End}_R(M/N)$ with $\varphi(f) = f^*$ for any $f \in S$. As M is quasi-projective, φ is an epimorphism. Thus $\text{End}_R(M/N) \cong S/\text{Ker } \varphi$. By Corollary 3.13, S is π -regular, and so is $S/\text{Ker } \varphi$. Therefore M/N is dual π -Rickart due to Lemma 3.5. \square

Corollary 3.16. *Let M be a quasi-projective duo module. If M is dual π -Rickart, then M/N is also dual π -Rickart for every submodule N of M .*

Corollary 3.17. *If M be a quasi-projective dual π -Rickart module, then so is $M/\text{Rad}(M)$ and $M/\text{Soc}(M)$.*

Proposition 3.18. *Let M be a dual π -Rickart module. Then every endomorphism of M with a small image in M is nilpotent.*

Proof. Let $f \in S$ with $\text{Im} f$ small in M . Then $\text{Im} f^n$ is a direct summand of M for some positive integer n . Also $\text{Im} f^n$ is small in M . Hence $f^n = 0$. \square

Corollary 3.19. *Let M be a dual π -Rickart discrete module. Then $J(S)$ is nil and $S/J(S)$ is von Neumann regular.*

Proof. Since M is discrete, by [13, Theorem 5.4], $J(S)$ consists of endomorphisms with small image. By Proposition 3.18, $J(S)$ is nil and again by [13, Theorem 5.4], $S/J(S)$ is von Neumann regular. \square

Theorem 3.20. *The following are equivalent for a module M .*

- (1) M is a dual π -Rickart module.
- (2) S is a generalized left principally projective ring and $f^n M = r_M(l_S(f^n M))$ for all $f \in S$ and a positive integer n .

Proof. (1) \Rightarrow (2) By Lemma 3.1, we only need to show that $f^n M = r_M(l_S(f^n M))$ for all $f \in S$. Since M is dual π -Rickart, for any $f \in S$, $f^n M = eM$ for some $e^2 = e \in S$ and a positive integer n . Thus $r_M(l_S(f^n M)) = r_M(l_S(eM)) = eM = f^n M$.

(2) \Rightarrow (1) Let $f \in S$. Since S is a generalized left principally projective ring, $l_S(f^n M) = Se$ for some $e^2 = e \in S$ and a positive integer n . By hypothesis, $f^n M = r_M(l_S(f^n M)) = r_M(Se) = (1-e)M$. Thus M is dual π -Rickart. \square

Corollary 3.21. *Let M be a module. Then M is dual π -Rickart if and only if $f^n M = r_M(l_S(f^n M))$ and $r_M(l_S(f^n M))$ is a direct summand of M .*

Theorem 3.22. *Let M be a dual π -Rickart module. Then the left singular ideal $Z_l(S)$ of S is nil and $Z_l(S) \subseteq J(S)$.*

Proof. Let $f \in Z_l(S)$. Since M is dual π -Rickart, $\text{Im}(f^n) = eM$ for some positive integer n and $e = e^2 \in S$. Then, by Lemma 3.1, $l_S(f^n) = S(1-e)$. Since $l_S(f^n)$ is essential in S as a left ideal, we have $l_S(f^n) = S$. This implies that $f^n = 0$ and so $Z_l(S)$ is nil. On the other hand, for any $g \in S$ and $f \in Z_l(S)$, according to previous discussion, $(gf)^n = 0$ for some positive integer n . Hence $1 - gf$ is invertible. Thus $f \in J(S)$. Therefore $Z_l(S) \subseteq J(S)$. \square

Proposition 3.23. *The following are equivalent for a module M .*

- (1) M is an indecomposable dual π -Rickart module.
- (2) Each element of S is either an epimorphism or nilpotent.

Proof. (1) \Rightarrow (2) Let $f \in S$. Then $f^n M$ is a direct summand of M for some positive integer n . As M is indecomposable, we see that $f^n M = 0$ or $f^n M = M$. This implies that f is an epimorphism or nilpotent.

(2) \Rightarrow (1) Let $e = e^2 \in S$. If e is nilpotent, then $e = 0$. If e is an epimorphism, then $e = 1$. Hence M is indecomposable. Also for any $f \in S$, $fM = M$ or $f^n M = 0$ for some positive integer n . Therefore M is dual π -Rickart. \square

Theorem 3.24. *Consider the following conditions for a module M .*

- (1) *S is a local ring with nil Jacobson radical.*
- (2) *M is an indecomposable dual π -Rickart module.*

Then (1) \Rightarrow (2). If M is a morphic module, then (2) \Rightarrow (1).

Proof. (1) \Rightarrow (2) Clearly, each element of S is either an epimorphism or nilpotent. Then M is indecomposable dual π -Rickart due to Proposition 3.23.

(2) \Rightarrow (1) Let $f \in S$. Then $f^n M = eM$ for some positive integer n and an idempotent e in S . If $e = 1$, then f is an epimorphism. Since M is morphic, f is invertible by [9, Corollary 2]. If $e = 0$, then $f^n = 0$. Hence $1 - f$ is invertible. This implies that S is a local ring. Now let $0 \neq f \in J(S)$. Since f is not invertible and M is morphic, f is nilpotent by Proposition 3.23. Therefore $J(S)$ is nil. \square

The next result can be obtained from Theorem 3.24 and [6, Lemma 2.11].

Corollary 3.25. *Let M be an indecomposable dual π -Rickart module. If M is morphic, then S is a left and right π -morphic ring.*

Acknowledgements. The authors would like to thank the referee(s) for valuable suggestions. The first author thanks the Scientific and Technological Research Council of Turkey (TUBITAK) for the financial support.

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